# ABERRATION WITH TWO EYES

by

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## Dedicated to H.-J.Treder

The calculation of the apparent image seen by a moving observer seems to be a problem of Special Relativity. Textbooks cite the fact, that the apparent image of a moving object, if small enough, is not sheared, but rotated and resized at best. However, one rarely finds a comparison with the non-relativistic case, i.e. with the corresponding phenomena in Newton's optics. That is, the effects of aberration – present in Newton's optics too – and of relativity (f.i. length contraction) are not separated. Secondly, the question of the apparent distance of the object is never considered as a stereoscopic problem. In the following, we will do just this and show how aberration can be defined for non-relativistic signals and how it is modified with respect to relativity; that the lack of shear on the apparent sphere is a relativistic effect, and that the images are sheared in the non-relativistic case; that the image is contracted or expanded with respect to its depth, depending on its apparent position, so that in three dimensions shear is usually present; that the three-dimensional shear of the stereoscopic images depends not only on the angles between apparent position and velocity, but also on the direction of the line through the two eyes.

It is always impressive to discuss with H.-J.Treder scientific topics, in particular if fundamental issues are raised. When I meet him he seems to me a living example for one of Einstein's aphorisms for Leo Baeck, that about the fascination through work. I wish him to continue that way and dedicate to him the topic of such a discussion.

# 1 The scene

An observer is moving uniformly $\vec{r} = \vec{v}t$ along a straight line through a world of objects which are assumed to be at rest relative to each other, thus defining a reference frame in which it is easy to calculate. We are interested in the apparent image of this world, seen by the observer at some instant $t = 0$ at the position $\vec{r} = 0$. In the rest frame of the objects, the apparent position of a point $\vec{s}$ in space is given by the normal vector

$$\vec{n} = \frac{\vec{s} - \vec{r}}{|\vec{s} - \vec{r}|} \ .$$

If the observer moves, this direction is shifted by aberration. We get a new apparent ray vector,

$$\vec{n}^* = \vec{n}^*[\vec{n}, \vec{v}] \ . \tag{1}$$

Throughout this article, it will not be normalized. In a Galilean world, with isotropic signal propagation in the rest frame of the objects (Newton's emanation hypothesis) and Huygens's additive composition law of velocities, we get simply

$$\vec{n}^* = \vec{n} + \frac{\vec{v}}{c}\ , \tag{2}$$

the constant $c$ denoting the signal propagation velocity. To effects of the order $c^2_{\text{sound}}/c^2_{\text{light}}$, wave-packets of sound or other particle-type signals are an appropriate model for this setting. We note the curious difference in aberration for particle-type signals and normals of wave-fronts present in the Galilean setting: The normals of wavefronts are not affected by the motion of an observer. Suppose a planar antenna for sound that measures by interference the wave front simultaneously. We would not observe any aberration. Only an ear that can measure the direction of an interference pattern in an undisturbed flowing medium would observe aberration – like the Fresnel telescope that is supposed not to disturb the freely flowing ether.

In a relativistic world, equation (2) is to be replaced by

$$\vec{n}^* = \vec{n} - \frac{\vec{v}(\vec{v},\vec{n})}{v^2} + \vec{v}\frac{(\frac{(\vec{n},\vec{v})}{v^2} + \frac{1}{c})}{\sqrt{1-\frac{v^2}{c^2}}}\ .$$

In this case, the map

$$\vec{n} \to \frac{\vec{n}^*[\vec{n}]}{|\vec{n}^*[\vec{n}]|}$$

is a conformal map on the unit sphere. Small apparent images in the moving system are only shifted, rotated (around the line of sight only) and resized with respect to the system at rest (fig.1), but not sheared [5, 1, 2]. In relativity, there is no difference between the direction of particles and that of normals of wavefronts. Wavefronts are tilted by the relativity of simultaneity, and their normals behave like the directions of particles. In relativity, the composition law of velocities can be derived from the laws of the propagation of light.

The two-dimensional apparent picture was complemented to a three-dimensional map by Scott and Viner [4]. The argument used the length of the light path from the object to the observer and compared the corresponding times in both systems of reference. Thus, the one-eye view was kept and improved by a time observation. We will follow the more natural way of the stereoscopic view – including only measurements of apparent angles – to obtain the apparent distances. Consequently, we assume the observer to see with two eyes, infinitesimally distant (fig. 2). This allows him to compare two infinitesimally different apparent normal vectors and conclude a parallactic distance. It is the aim of this paper to calculate this new map of the three-dimensional space on itself and to discuss its properties and to compare acoustical and optical appearance.

## 2 The map

We denote the direction of the velocity with $\vec{w}$ to get $\vec{v}/c = \beta\vec{w}$, and the Lorentz factor by $\gamma = 1/\sqrt{1-\beta^2}$. Let us assume we have the normal $\vec{n}[\vec{s},\vec{r}]$ for an eye (or ear, resp.) at the

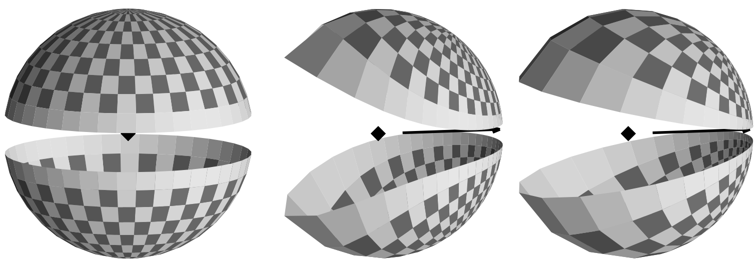

Figure 1: The apparent sphere for the moving observer
The observer at the center is moving to the right. On the left, the observer is at rest and sees the undisturbed sphere. For the moving observer, the points of the apparent sphere are shifted in the direction of the apex as indicated by the coordinate net. In the middle, we see the non-relativistic, on the right the relativistic case. The velocity is in both cases $\beta = 0.8$. The most conspicuous property of the relativistic map is the flatness of the circles of constant latitude. This indicates a projective and still more conformal map. In the non-relativistic case, the circle of former constant latitude are not plane. More general pictures can be found in [3].

general position $\vec{r}$. We then get for the first eye at $\vec{r} = 0$ the normal $\vec{n}_1 = \vec{n}[\vec{s}, 0]$ and for the second eye at $\vec{r} = \mathrm{d}\vec{a}$

$$\vec{n}_2 = \vec{n}_1 + \mathrm{d}\vec{a}\,\frac{\mathrm{d}\vec{n}}{\mathrm{d}\vec{r}}\,|_{\text{at }\vec{r}=0} = \vec{n}_1 - \frac{1}{|\vec{s}|}(\mathrm{d}\vec{a} - \vec{n}(\vec{n}, \mathrm{d}\vec{a})) \quad .$$

Both normals are subject to the transformation, eq.(1), the distance $\mathrm{d}\vec{a}$ to the second eye may be modified by this transformation too (in the relativistic case only). The vector $\mathrm{d}\vec{a}$ and its component perpendicular to the normal, $\mathrm{d}\vec{a} - \vec{n}(\vec{n}, \mathrm{d}\vec{a})$ are both infinitesimal. We obtain the map by the following steps.

1. Choose the position $\mathrm{d}\vec{a}^*$ of the eyes.

2. Transform this position into the rest frame of the objects,

$$\mathrm{d}\vec{a} = \mathrm{d}\vec{a}^* + (\gamma - 1)\vec{w}(\vec{w}, \mathrm{d}\vec{a}^*) \ .$$

   The two eyes measure simultaneously in their rest frame. The factor $\gamma$ makes the difference to acoustics, where it has to be put equal to 1.

3. Get the apparent position $\vec{n}$ in the rest frame of the objects. and apply the aberration formula $\vec{n}^* = \vec{n}^*[\vec{n}, \vec{v}]$ to get the apparent position in the rest frame of the observer.

4. Get the difference of the apparent positions,

$$-s\ \mathrm{d}\vec{n} = \mathrm{d}\vec{a} - \vec{n}(\vec{n}, \mathrm{d}\vec{a}).$$

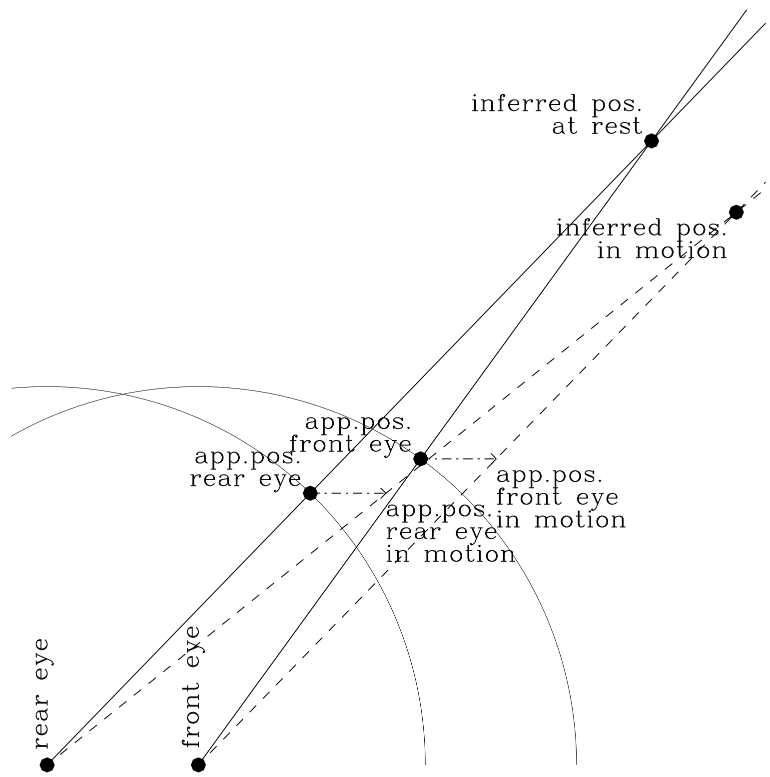


Figure 2: Stereoscopic aberration.

The figure show schematically the procedure of inferring the position of an object for two observers in relative motion. For staying in two dimensions, we chose the case to two eyes looking sideward with respect to the relative motion. Both observers infer a position. The difference in the positions is due to the relative motion of the observers.

5. Transform into the rest system of the observer,

$$d\vec{n}^* = d\vec{n} + (\gamma - 1)\vec{w}(\vec{w}, d\vec{n}) \ .$$

6. Solve the system

$$\vec{s}^* = \lambda \vec{n}^* = d\vec{a}^* + \mu \ (\vec{n}^* + d\vec{n}^*) \ .$$

Its formal solution for $\mu$ is found by the cross product with $\vec{n}^*$:

$$(d\vec{a}^* \times \vec{n}^*) + \mu \ (d\vec{n}^* \times \vec{n}^*) = 0 \ .$$

The formal solution exists only in the case where the three vectors $d\vec{a}^*$, $\vec{n}^*$, and $d\vec{n}^*$ are coplanar. If they are not, the two equations for $\vec{s}$, $\lambda$, and $\mu$ have no common solution. That is, after aberration the two rays are no more in a common plane. We obtain this case, if the line of sight, the velocity and the eye line are not coplanar.

If the distance of the eyes is infinitesimal, the error can be neglected to first order. We choose

$$\vec{s}^* = -\vec{n}^* \cdot \frac{(d\vec{a}^* \times \vec{n}^*)(d\vec{n}^* \times \vec{n}^*)}{(d\vec{n}^* \times \vec{n}^*)(d\vec{n}^* \times \vec{n}^*)} = \ \vec{n}^* \ \frac{(d\vec{a}^*, d\vec{n}^*)(\vec{n}^*, \vec{n}^*) - (d\vec{a}^*, \vec{n}^*)(d\vec{n}^*, \vec{n}^*)}{(d\vec{n}^*, d\vec{n}^*)(\vec{n}^*, \vec{n}^*) - (d\vec{n}^*, \vec{n}^*)(d\vec{n}^*, \vec{n}^*)} + O[d\vec{a}^*] \ . \quad (3)$$

This formula has to be evaluated in the cases differing in the angle between the eye line $d\vec{a}$ and the velocity $\vec{v}$. The resulting map is the product of a redirection $\vec{n}^*[\vec{n}, \vec{v}]$ (not depending on the distance and on the direction of the eye line) and of a gauge of the distance depending on the the direction of the eye line too. Formula eq.(3) is homogeneous in $d\vec{a}^*$. To evaluate it, we can normalize $d\vec{a}^*$ and forget that it is infinitesimal originally.

# 3 Particular cases

Throughout this section, we denote the direction of the velocity with $\vec{w}$, and choose for the direction cosines

$$\xi = (\vec{n}, \vec{w}) \ , \quad \eta = (\vec{n}, d\vec{a}^*) \ , \quad \zeta = (\vec{w}, d\vec{a}^*) \ .$$

In addition, we use the abbreviations

$$\beta^* = \beta\gamma + \xi(\gamma - 1) \ , \quad \eta^* = \eta + \xi\zeta(\gamma - 1) \ .$$

We obtain

$$\begin{aligned} d\vec{a} &= d\vec{a}^* + (\gamma - 1)\zeta\vec{w} && (4) \\ -s \ d\vec{n} &= d\vec{a}^* + (\gamma - 1)\zeta\vec{w} - \eta^*\vec{n} && (5) \\ \vec{n}^* &= \vec{n} + \beta^*\vec{w} && (6) \\ -s \ d\vec{n}^* &= d\vec{a}^* + (\gamma - 1)\zeta\vec{w} - \vec{n}\eta^* + (\gamma - 1)\vec{w}(\zeta\gamma - \xi\eta^*) && (7) \end{aligned}$$

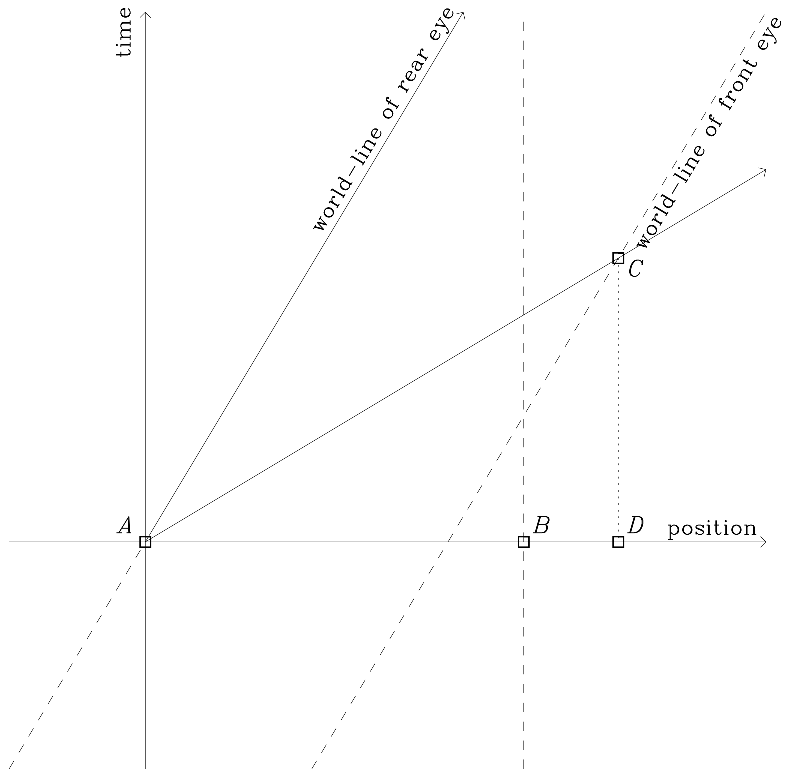


Figure 3: Length contraction and eyes distance.

We show the word-lines of two eyes at rest and two eyes moving – both pairs with the same proper distance – in a space time diagram. Two events simultaneous in the rest-frame of the moving eyes are $O$ and $C$. Their distance $OD$ in the rest-frame of the objects is larger than their proper distance $OB$. Evaluated from the latter frame, the front eye measures later than the rear one. This produces no effect, because the objects are assumed to be at rest. Otherwise, retardation effects are superposed.

### 3.1 Looking sideward

We assume the eye line $\mathrm{d}\vec{a}^*$ to be parallel to the velocity $\vec{v}$ (i.e., $\mathrm{d}\vec{a}^* = \vec{w}$, $\zeta = 1$, $\eta = \xi$, $\eta^* = \xi\gamma$). For a moving train, this is just looking quietly out of the window of a wagon. We obtain the map

$$\vec{s}^* = s\ \frac{\vec{n} + (\xi(\gamma - 1) + \beta\gamma)\vec{w}}{\gamma^2(1 + \beta\xi)}\ .$$

This map has some peculiar properties. Any sphere (referring to objects rest) around the observer is transformed into a another (smaller) concentric sphere, the surface being shifted to the pole of motion, in a conformal way. The latter is known, the former is new. The sphere $x^2 + y^2 + z^2 = r^2$ is mapped onto $\gamma^2(x^{*2} + y^{*2} + z^{*2}) = r^2$. The true image of a infinitesimal cube facing the observer is a quadratic prism, facing the observer again with a square with a size depending on apparent position, and a depth independent on position, shortened by the Lorentz factor $\gamma$. Shear is present in the general case, but not in the projection onto the apparent sphere. The old question of the visibility of Lorentz contraction can be answered positively in this case. However, we do not see the Lorentz contraction of the moving object, but the elongation of the eye distance in the objects frame due to a combination of Lorentz contraction and change in simultaneity (fig. 3).

To compare with non relativistic aberration, we have to put $\gamma = 1$ in the formulas, eq.(1-3), and obtain

$$\vec{s}^* = s\frac{\vec{n} + \vec{v}}{1 + \beta\xi}\ .$$

Here, the sphere $x^2 + y^2 + z^2 = r^2$ is mapped onto an ellipsoid of the form

$$\gamma^2 x^{*2} + y^{*2} + z^{*2} = \gamma^2 r^2\ .$$

The ellipsoid is contracted in the direction of motion if compared with the other directions. Figure 4 shows the non-relativistic (acoustical) and the relativistic (optical) image of a sphere for an observer in the center, moving with a speed corresponding to $\beta = 0.8$. Figure 5 shows the image of a cylindrical channel. The qualitative similarity for acoustics and optics is evident, in spite of the high velocity used to construct the image.

### 3.2 Looking on each point individually

We now assume the eyes connecting line $\mathrm{d}\vec{a}^*$ to lie perpendicular to the direction of motion ($\zeta = 0$). In addition, we turn the eyes connecting line in such a way that the object is always on the optical axis, i.e. $(\vec{n}^*, \mathrm{d}\vec{a}^*) = 0$, $\eta = \eta^* = 0$. The map is the particularly simple standard map now. We obtain

$$-s\mathrm{d}\vec{n}^* = \mathrm{d}\vec{a}^*\ ,\quad \mu = -1\ .$$

$$x^* = \gamma(x - \beta\sqrt{x^2 + y^2 + z^2})\ ,\quad y^* = y\ ,\quad z^* = z\ .$$

The image of the $x$-$z$ plane is shown in fig.6 (lower part). In the non-relativistic case, we

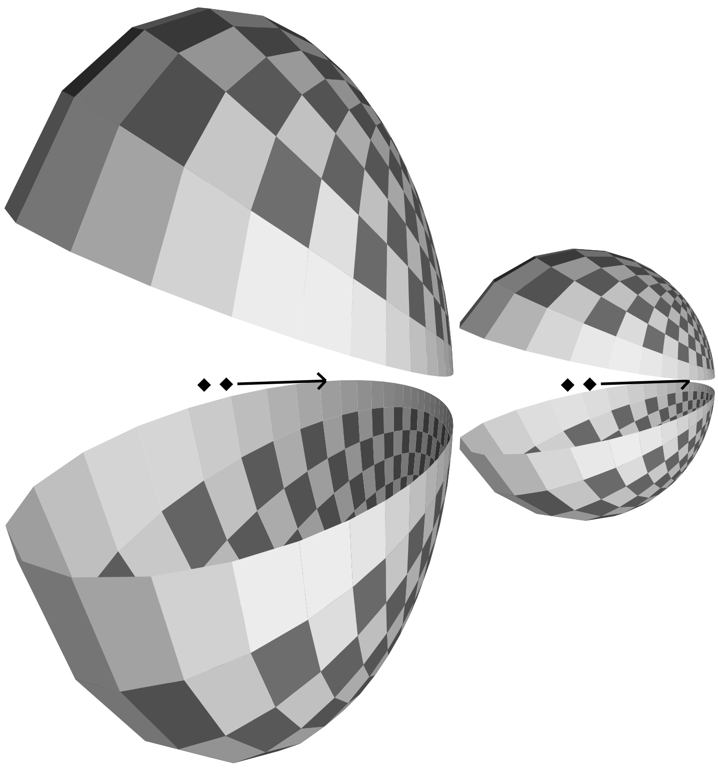

Figure 4: The true image of a sphere for the moving observer, looking sideward
In the relativistic case (right part) the true image of a sphere around the observer at rest which the moving observer obtains is again a sphere, but contracted by the Lorentz factor $\gamma$. This is the pure effect of the Lorentz contraction of the eye distance in the frame at rest to the moving frame. In the non-relativistic case (left part) the true image is a rotation ellipsoid stretched perpendicular to the direction of motion by the Lorentz factor $\gamma$. Surprisingly, this factor has a role in a non-relativistic situation too. The deformation of the sphere renders the images of the circles of constant latitude plane again. The velocity is in both parts $\beta = 0.8$.

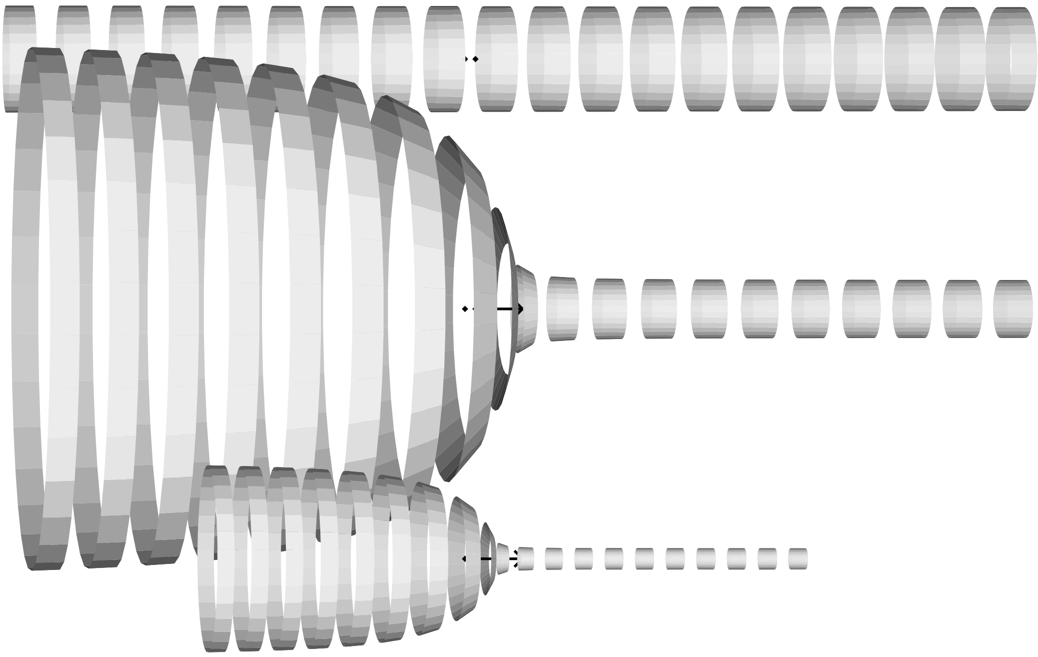

Figure 5: The channel for the moving observer, looking sideward
The channel at rest is given by a series of rings (upper part). The relativistic case is shown in the lower part. In forward direction, the channel seems to be narrower (factor $(1-\beta)$), backward, the channel seems wider (factor $(1+\beta)$). In both cases the channel seems to be contracted by the Lorentz factor $\gamma$. Again, this is the effect of the Lorentz contraction of the eyes distance. The non-relativistic case is shown in the middle. Again, in forward direction the channel seems narrower, this time with the factor $(1+\beta)^{-1}$. Backward, the channel seems wider by the factor $(1-\beta)^{-1}$. No contraction is seen. ($\beta = 0.8$.)

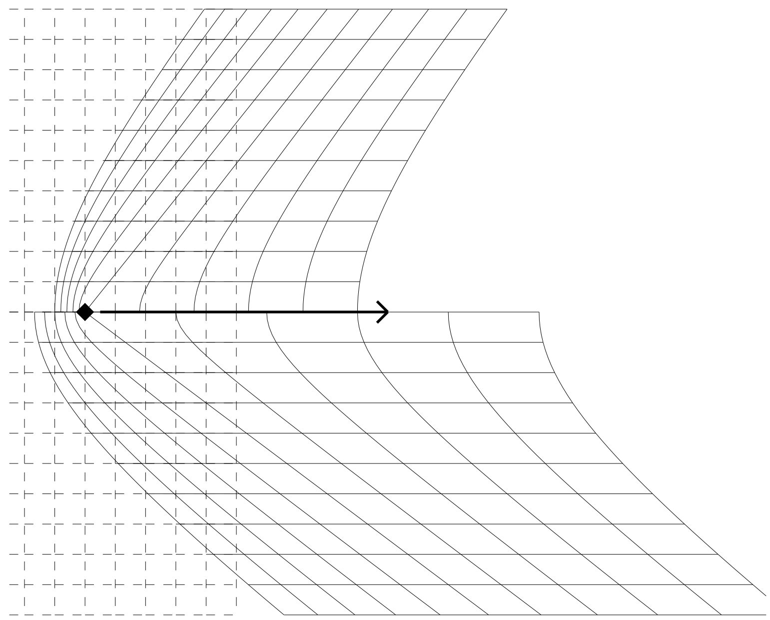

Figure 6: The standard map of the plane
We show the distortion of a rectangular grid by the standard aberration map. The case of Einstein's composition of velocities is seen in the upper part, the case of Huygens's composition law in the lower part.

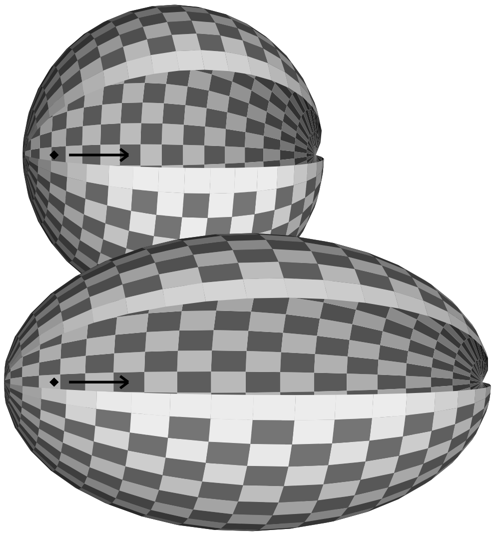

Figure 7: The sphere for the moving observer, turning the head
We show the distortion of a sphere by the standard aberration map. The case of Einstein's composition of velocities is seen in the upper part, the case of Huygens's composition law in the lower part.

have only to substitute 1 for $\gamma$ (upper part). The sphere $x^2 + y^2 + z^2 = r^2$ is mapped to

$$(\frac{x^*}{\gamma} - \beta)^2 + y^{*2} + z^{*2} = r^2 \ .$$

This is the map calculated by Scott and Viner [4]. Here, the original sphere is shifted and elongated into an ellipsoid in the apex direction. In the non-relativistic case, we have to substitute 1 for $\gamma$ again, so the sphere is still shifted, but no more elongated.

### 3.3 Looking forward

The eye line is still perpendicular to the direction of motion, but shall *not* follow the object ($\zeta = 0$, $\eta^* = \eta$). The observer is looking out of the front window. We begin with

$$-s\mathrm{d}\vec{n}^* = \mathrm{d}\vec{a}^* - \eta\vec{n} - \vec{w}\eta\xi(\gamma - 1) \ .$$

The scalar products are

$$s^2 \ (\mathrm{d}\vec{n}^*\mathrm{d}\vec{n}^*) = 1 - \eta^2 + (\gamma^2 - 1)\xi^2\eta^2 = \begin{array}{ll} 1 - \eta^2 & \text{for } \ \gamma = 1 \\ 1 - \eta^2 + \xi^2\eta^2\beta^2\gamma^2 & \text{for } \ \gamma^2 = 1 + \beta^2\gamma^2 \end{array}$$

$$-s \ (\mathrm{d}\vec{n}^*\vec{n}^*) = -\xi\eta(\beta\gamma^2 + \xi(\gamma^2 - 1)) = \begin{array}{ll} -\xi\eta\beta & \text{for } \ \gamma = 1 \\ -\xi\eta\beta\gamma^2(1 + \beta\xi) & \text{for } \ \gamma^2 = 1 + \beta^2\gamma^2 \end{array}$$

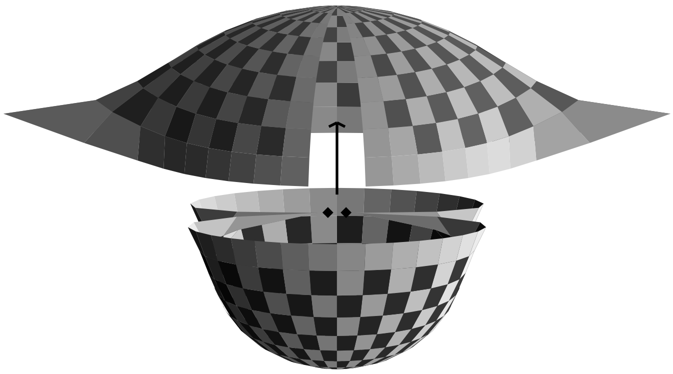

Figure 8: The sphere for the moving observer, looking forward
The velocity is directed upward in this picture. The map is singular because of the zero in the nominator of eq.(9). ($\beta = 0.14$.)

$$(\vec{n}^*\vec{n}^*) = 1 + 2\xi\beta^* + \beta^{*2} = \begin{array}{ll} 1 + 2\xi\beta + \beta^2 & \text{for } \ \gamma = 1 \\ (1+\xi\beta)^2\gamma^2 & \text{for } \ \gamma^2 = 1 + \beta^2\gamma^2 \end{array}$$

$$-s\ (\mathrm{d}\vec{a}^*\mathrm{d}\vec{n}^*) = 1 - \eta^2$$

$$(\mathrm{d}\vec{a}^*\vec{n}^*) = \eta$$

For $\gamma = 1$, we obtain

$$\mu = 1 + \frac{\beta\xi\eta^2(1+\beta\xi)}{(1-\eta^2)(1+2\xi\beta+\beta^2) - \beta^2\xi^2\eta^2} \tag{8}$$

For $\gamma^2 = 1 + \beta^2\gamma^2$, we obtain

$$\mu = 1 + \frac{\beta\xi\eta^2}{(1-\eta^2)(1+\xi\beta)^2} \tag{9}$$

These maps are topologically different from the map in the former two cases. For both composition laws, the map becomes singular at $\eta = \pm 1$ (fig.8). We obtain the remarkable result that the configurations seen by an observer with two eyes strongly depends on the direction of the eye line. This is already a non-relativistic effect, present for galilean aberration too. Figure 8 shows the image of a sphere for a moving observer, looking forward. The image is calculated for the non-relativistic formula and only with the moderate speed of $\beta = 0.14$ in order to keep the singular feature inside the figure. The relativistic figure does not differ essentially. The zero of the nominator in equation (8) or equation (9) determines this singularity of the map, present already for very small velocities. Figure 9 repeats the image of the channel, this time for the eye line orthogonal to the velocity, i.e. for the man in the cockpit of the train. The velocity is again $\beta = 0.8$.

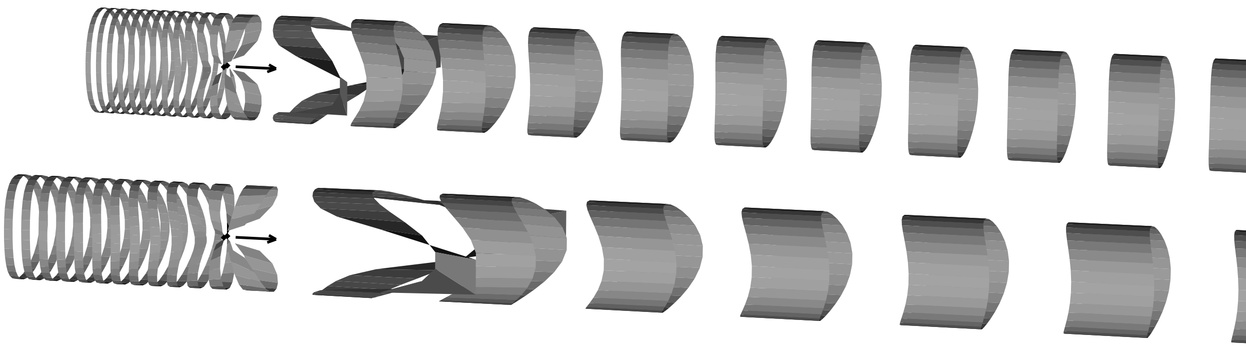

Figure 9: The channel for the moving observer, looking forward
The relativistic case is shown in the lower part. In the far, the channel is not wider or narrower, but it is stretched by the factor $\gamma(1+\beta)$ in forward and contracted by the factor $\gamma(1-\beta)$ in backward direction. In the non-relativistic case (shown in the upper part), the stretching in forward direction is given by the factor $(1+\beta)$, the contraction in backward direction by the factor $(1-\beta)$. In both cases, the channel is strongly deformed locally. Both maps contain a singularity.

## 3.4 Conclusion

The one-eye map of the apparent spheres can be generalized to the stereoscopic case of two eyes. The result is the true three-dimensional image of a world – with all objects except the observer at rest with respect to each other – for an observer moving through this world. If an object is moving uniformly in this world, we have to use a transformation from the rest system of that object to that of the observer, so no further generalization can be expected. Three facts are remarkable.

- First, the image of the world for a moving observer looks different depending on the direction of his meridian. The images contain singularities except for some peculiar prescriptions for the position of the eyes.

- Second, all peculiarities are present in the non-relativistic case too, if we accept there a finite signal propagation velocity. This velocity is assumed here to be isotropic in the reference of the objects. Any motion of an object with respect to this reference frame will then add further complications.

- Third, the pure length contraction remains hidden in the more complicated outlay of the general aberration. The definition of length contraction requires simultaneous readout of position, and this is not so simple to prepare, taking into account the finite velocity of light.

Any position measurement by triangulation will be determined to first order by aberration and only to second order corrected by length contraction. For small $\beta$ (referring to the singal propagation velocity in question) an observer hearing stereoscopically will experience nearly the same picture as an observer seeing stereoscopically. The differences will always be second order in $\beta$.

We always compared optics withs acoustics in a Galilean frame. Of course, we may generalize acoustics to include relativity, i.e. Lorentz transformations in general and Einsteins non-additive composition of velocities and Lorentz contraction in particular. Because of the small value of $c^2_{\mathrm{sound}}/c^2_{\mathrm{light}}$, the differences will be marginal. In both cases, sound propagation breaks relativity, if no medium of propagation is taken into account.

The transition to the problem of the moving object and an observer at rest is trivial in the relativistic case: There is no difference in the appearance. In the non-relatistic case, we have to calculate with the time of flight of the signal to obtain the position of the object at the instant, when the signal reaching the observer at $t = 0$ starts. This time of flight will be different if the signal is a wave in a medium propagating isotropically in the rest frame of the observer or a particle propagating isotropically in the rest frame of the object.

I thank H.-J.Treder for the merry discussion.

# References


[1] Penrose,R. (1959): The apparent shape of a relativistically moving sphere, *Proc.Camb.Phil.Soc.* **55**, 137.

[2] Ruder,H., Ruder,M. (1994): *Die Spezielle Relativitätstheorie*, Vieweg Braunschweig.

[3] Scott,G.D., van Driel,H.J. (1970): Geometrical appearances at relativistic speeds, *Amer.J.Phys.* **38**, 971.

[4] Scott,G.D., Viner,M.R. (1965): The geometrical appearance of large objects moving at relativistic speeds, *Amer.J.Phys.* **33**, 534.

[5] Terrell,J. (1959): Invisibility of the Lorentz contraction, *Phys.Rev.* **116**, 1041.