Multifactor cosmological models

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Abstract

We review the phenomenological properties of Kaluza-Klein cosmologies with an open number of compactified factor spaces. There exist general arguments for the necessity of positive internal curvatures to produce a dynamic compactification. The general procedure to get exact solutions as to classical minisuperspace evolution as to the Wheeler-deWitt equation is considered.

1 Introduction

Multidimensional cosmological models are a phenomenological consequence of multidimensional field theories. Three-dimensional phenomenology requires the additional dimensions to form hidden factors of the space. The open question is the dynamical importance of the additional factors of the space, the question, if and how their size changes with time. At present time the factor spaces are hidden macroscopically by their small size. This size might change in the early history of the universe, but in recent times (primordial nucleosynthesis and later) this size should be nearly constant in order to save the observations of the invariability of the fundamental constants [27, 30, 31], or the first law of thermodynamics, or the Laplace law of light intensity, or the laws of atomic structure [17]. The effects of higher

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dimensions should be expected in the early universe, where we should assume all dimensions to be of the same size as an expression of unbroken symmetry. The breakdown of symmetry below the Curie temperature of the vacuum should reveal the differences, i.e. should factor the additional dimensions off the conventional 3-space and reduce their size to microscopic magnitudes, possibly parallel to an inflation of the 3-space [19, 24, 29, 32]. For simplicity, one usually assumes the additional dimension to form one unique factor space, with different topologies, but nearly isotropic by itself. The configuration of the factorwise isotropic cosmological model is then given by two expansion parameters. We might conjecture that such a model will miss eventually chaotic behaviour and other instabilities typical for higher-dimensional configuration and phase spaces [13, 33]. That is the reason why one should think about multifactor models, where the additional dimensions form more than one hidden factor of the space.

2 The particle analogy

If the D-dimensional space is split into individual d_j-dimensional factors ($D = \sum_j d_j$), a homogeneous isotropic universe has to be isotropic only factorwise. Instead of a line element of the form

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - R^2[t]\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\sigma^2)_{(D-1)-\text{dimensional unit sphere}}\right) , \quad (1)$$

we write

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \sum_{j=0}^{\alpha}(R_j[t])^2\frac{\sum_{i=1}^{d_j}(\mathrm{d}x^i)^2}{(1+\frac{k_j r_j^2}{4})^2} . \quad (2)$$

That is the line element of a world with $\alpha+1$ factors, each homogeneous and isotropic, and each with their individual expansion parameter, or curvature radius, $R_j[t]$, and curvature index, k_j. Each factor adds one term, the first being always the conventional 3-space ($d_0 = 3$). From the corresponding point of view, these cosmological models generalize the simplest anisotropic homogeneous solutions of General Relativity.

In our model, we have to assume a phenomenological matter tensor generalizing the ideal fluid by allowing individual pressure values p_j for each factor space. The energy-momentum tensor T^a_b is diagonal again, but the pressure might differ in directions, which belong to different factors of space. Microscopically, that corresponds to particle interactions, which do not allow to redistribute the kinetic energies between the individual factor spaces, but only internally in each factor. This produces a kind of constrained equilibrium which allows for the different pressure values.

The continuity equation in such a model is given by

$$\mathrm{d}\varepsilon = -\sum_{j=0}^{\alpha} d_j \frac{\mathrm{d}R_j}{R_j}(\varepsilon + p_j) \ . \tag{3}$$

The energy density ε is a density in a D-dimensional space. To be interpreted in 3-dimensional terms, it has to be integrated over all the additional dimensions. The ordinary continuity equation (i.e. the first law of thermodynamics) is bound to

$$\sum_{j=1}^{\alpha} d_j \frac{\mathrm{d}R_j}{R_j} p_j = 0 \ . \tag{4}$$

This has to be read as a condition on the equation of state in the additional space factors or on their expansion rates. Correspondig restrictions to the equations of state are difficult to imagine, so these conditions should concern the expansion rates. The first law of thermodynamics should be understood as proof of the extreme smallness of internal expansion rates today. This is backed by the fact that we do not observe any change in fundamental constants, which would be produced by non-vanishing internal expansion rates [27, 30].

Just as in the standard model, we describe the matter as a mixture of special barotropic components, which have a linear equation of state,

$$p_j = (\frac{m_j}{d_j} - 1)\varepsilon \ . \tag{5}$$

With the continuity equation, eq. (3), we get the integral

$$\varepsilon = M_{m_0 m_1 \ldots m_\alpha} \frac{R_0^{m_0}[0] R_1^{m_1}[0] \ldots R_\alpha^{m_\alpha}[0]}{R_0^{m_0} R_1^{m_1} \ldots R_\alpha^{m_\alpha}} \ . \tag{6}$$

Such barotropic equations of state describe “cold” matter in the sense, that temperature does not enter. Any such matter component is characterized by the family (m_0, m_1,..., m_α) of indices[18]. In analogy to the 4-dimensional world we can include in this form the curvatures of the individual factor spaces and the cosmological constant [5]. The cosmological constant is characterized by the index family $(0, 0, \ldots, 0)$, the curvature of the k-th factor by ($m_i = 0$ for $i \neq k$, and $m_k = 2$). The important case of vanishing trace of the energy-momentum tensor will be called superradiation. It is given by

$$\sum_{i=0}^{\alpha} m_i = 1 + \sum_{i=0}^{\alpha} d_i \ . \tag{7}$$

Examples for different matter components in the case of two space factors is shown in table 1. This is the place to remind that different models of the vacuum produce vacuum components, which do not coincide dynamically with the cosmological constant.

The main advantage of reducing the matter components to equations of state like eq.(5) is to map the cosmological evolution onto the motion of a particle in an formal $(\alpha+1)$ dimensional space which carries a scalar potential [5, 8, 15, 34]. That is because the Einstein equations take the form

$$\begin{aligned} (\sum_{j=0}^{\alpha} d_j \frac{\dot{R}_j}{R_j})^2 - \sum_{j=0}^{\alpha} d_j (\frac{\dot{R}_j^2}{R_j^2} - (d_j - 1)\frac{k_j}{R_j^2}) &= 2\kappa\varepsilon \ , \\ \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\dot{R}_i}{R_i}) + \frac{\dot{R}_i}{R_i}\sum_{j=0}^{\alpha} d_j \frac{\dot{R}_j}{R_j} - (d_i - 1)\frac{k_i}{R_i^2} &= \kappa(p_i + \frac{\varepsilon - \sum_j d_j p_j}{D-1}) \ , \end{aligned}$$

where p_j and ε are given by eqs.(3) and eqs.(6). In the case of the density being the sum of cold components, this can be transformed to

$$m_{ij}\frac{\mathrm{d}^2\xi^j}{\mathrm{d}\tau^2} = \frac{\partial\Phi}{\partial\xi^i} \ . \tag{8}$$

The first integral is simply

$$m_{ij}\frac{\mathrm{d}\xi^i}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^j}{\mathrm{d}\tau} = 2\Phi \ . \tag{9}$$

Here, the formal coordinates ξ_j are

$$\xi^j = \ln \frac{R_j}{R_{j0}} \ , \tag{10}$$

Table 1: Matter components in a two-factor cosmological model

Indices	Interpretation	Source
m, n		
$m+n=d+4$	superradiation	[5]
$m=3, n=d+1$	dust-like superradiation	[5]
$m=4, n=d$	radiation-like superradiation	[5]
	radiation at low temperatures	
$m=0, n=d+4$	vacuum by Candelas & Weinberg	[12]
	contribution of Casimir energy	
$m=d+4, n=0$	vacuum by Moss	[26]
$m=3\frac{d+4}{d+3}, n=d\frac{d+4}{d+3}$	vacuum by Sahdev	[28]
	radiation at high temperatures	
$m=6, n=2d$	ultrastiff fluids by Zeldovich	[5]
$m=0, n=2d$	variant of string vacuum	[15]
	monopole effects	
$m=6, n=0$	variant of string vacuum	[15]
$m=2, n=0$	curvature of ordinary space	
$m=0, n=2$	curvature of additional space factor	
$m=0, n=0$	cosmological constant	

the formal time is

$$\mathrm{d}\tau = \mathrm{d}t \exp(-d_j \xi^j) \ . \qquad (11)$$

The potential is given by

$$\Phi = \frac{8\pi G}{c^4} \varepsilon \exp(2 d_j \xi^j) \ , \qquad (12)$$

and the formal mass matrix is the minisuperspace metrice

$$m_{ik} = d_i d_k - d_i \delta_{ik} \ , \quad m^{ik} = \frac{1}{D-1} - \frac{1}{d_k} \delta^{ik} \ . \qquad (13)$$

We now intend to determine the conditions which are necessary for the existence of solutions with negligible expansion rate of the supplementary space factors. To this end, we normalize the initial expansion parameters to be $\xi^j[t_0] = 0, \quad j = 0, \ldots, \alpha$. The time ist normalized by the Hubble age, i.e. $\dot{\xi}^0[t_0] = H_0$. The deceleration parameter is of the order of $\ddot{\xi}^0[t_0] = O(H_0^2)$. If the expansion rate of the internal space factors is much less than the Hubble factor, we have

$$\dot{\xi}^j[t_0] = o(H_0), \quad \ddot{\xi}^j[t_0] = o(H_0^2) \ . \qquad (14)$$

If we substitute this into the evolution equations (8) and (9), we get

$$\Phi[\xi[t_0]] = O(H_0^2) \ , \quad \frac{\partial \Phi}{\partial \xi^j}[\xi[t_0]] = O(H_0^2) \ . \qquad (15)$$

This implies

$$\begin{aligned} \frac{8\pi G}{3c^2} \sum_{\text{all components}} M_{m_0,\ldots,m_\alpha} &= O(H_0^2) \ , \\ \frac{8\pi G}{3c^2} \sum_{\text{all components}} m_i M_{m_0,\ldots,m_\alpha} &= O(H_0^2) \ , \quad i = 0, \cdots, \alpha \ . \end{aligned}$$

We got $\alpha+2$ conditions on the matter components which enclose cosmological constant and curvatures. The number of internal space factors is α. If we want to ensure their small size by large curvature, we have α formal matter components of large negative value, which have to be compensated in order to allow for slow internal expansion rates. Because we found $\alpha+2$ compensation conditions, we have to invent two other matter components as large as the curvatures. In order to circumvent this problem, one might opt for small size not by curvature, but by periodicity (i.e. torus topologies). This would be a requirement external to the problem, however.

3 Compactification and contraction

The cosmological model which we considered can describe the evolution only after the factorization of the D-dimensional space. This factorization is a symmetry breakdown, which also produces the finiteness (compactness) of the additional space factors. It is called compactification. Different models exist for this process. We can classify them in two classes: If the compactified factors are small from the very beginning, we call them spontaneous compactification. The program to get the small sizes by quasi-classical evolution as described above, we call dynamical compactification. This contraction of the internal space factors could and should be accompanied by the inflation of external size. This picture of dynamical compactification begins after the factorization with a state of expansion more or less common for all space factors (point A in fig.1),

$$\xi_0' = \xi_1' = \ldots = \xi_\alpha' \ . \tag{16}$$

In the final (late-time) state the expansion of the internal factors should have ceased (point C in fig.1),

$$\xi_1' = \ldots = \xi_\alpha' = 0 \ . \tag{17}$$

During this transition, the sizes of the internal space factors have to shrink to microscopic (much smaller than Heisenberg length) scale. The model will achieve this only by passing a point B in the hodograph, which lies in the lower half-plane. The formal acceleration vector (i.e. the tangent to a curve in the hodograph) has to point into this direction. For the equations of state shown in table 1 this is possible only for negative contributions to the total density ($M_{m_0,m_1} < 0$), and we get negative contributions only by positive curvatures, if we do not want to invent ad hoc exotic (antigravitating!) matter. A large internal curvature, however, produces the compensation problem just considered for late times. Then the internal expansion rate shall be negligible (point C in the hodograph). The large curvature has to be compensated in this stage, and just before it should be overcompensated, so that the fast contraction can be decelerated.

If the matter components have positive pressure in the internal space factors, the contraction of these yields an effective energy production from the 3-dimensional point of view, because 3-dimensional energy conservation

Figure 1: Hodograph of a two-factor model

The diagram shows the formal expansion rates of the external and the internal space factor. Common expansion in the time just after the factorization (i.e.compactification) is described by a point A on the diagonal. Stationarity of the internal space factor in late times is given by a point C on the $\dot{\xi}_0$ axis. The transition from A to C has to contain a fast and extreme contraction of the internal factor. Therefore, the path in the hodograph has to pass a point B in the lower half-plane. The tangential directions for the matter components in table 1 are shown in the lower left (positive contributions to the matter).

requires eq.(4). In addition, perturbations can acquire mass from the 3-dimensional point of view and present themselves as dark matter today [1, 8, 16].

4 Solutions to perfect fluid models

The equations of motion for the quasi-mechanical problem have exact solutions in several particular cases, all depending on the formal potential $\Phi[\xi]$. If the latter depends only on one linear combination of the coordinates ξ_k,

$$\Phi[\xi] = \sum_k M_k \exp(a_k \sum_j \omega_j \xi^j) \ , \tag{18}$$

we may choose this combination as one of a set of new coordinates and separate the system. This has been used for various combinations of fluid components, including curvatures and cosmological constant [5, 6, 9]. In particular, this case contains all models with a one-component potential (one-component fluid, one curvature, or cosmological constant), cited already in table 1.

Some reductions are also possible in the case where the quasi-potential depends on two independent linear combinations of the coordinates,

$$\Phi[\xi] = \sum_k M_k \exp(a_{1k} \sum_j \omega_{1j} \xi^j + a_{2k} \sum_j \omega_{2j} \xi^j) \ . \tag{19}$$

This case will cover any two-component fluid model [2, 9]. As any other model too, our approach will describe “the real situation” only approximately. The desire to get exact solutions should be questioned for that reason.

The integrability of our particle model has been considered. A sufficient condition is derived by the analogy to Toda-like systems for potentials including curvature only [3, 21].

Changing equations of state are represented in our model by changes in the composition of the fluid. It is the experience from conventional cosmological models, that curvature, cosmological constant, and mainly one fluid component, at most two, determine the evolution of the model. The transition from radiation-dominated to matter-dominated universe, for instance, is that fast, that we might represent this transition as a usual phase transition. Changing equations of state are used in [14]. They fit into our particle model, if the equation of state may be integrated to get a component of our formal potential.

The Wheeler-deWitt equation for the quasi-classical model is equal to the corresponding Schrödinger equation for zero formal energy. The operator of the kinetic energy, however, is not positive definite, because the minisuperspace metric is pseudoeuclidean [22, 7, 21]. In a covariant description in the minisuperspace (including freedom in the time shift) we have to write a covariant Hamiltonian, which implies a unique factor ordering [23].

5 Multifactor cosmological models with scalar field

Because multifactor cosmological models with perfect fluid matter source give us only a phenomenological model of cold matter, we have to fit a scalar field in the scheme, too. These models should describe the transition processes expected in the symmetry breakdown more appropriately.

Therefore, we consider a cosmological model with $\alpha + 1$ ($\alpha > 0$) Einstein spaces, containing a homogeneous, minimally coupled scalar field $\varphi[t]$ as a matter source and a cosmological constant Λ. The kinetic term of the scalar field does not change the form of the particle model. The potential of the scalar field is also added, but is only in a special case of the exponential form produced by the simple barotropic equations of state of the phenomenological approach.

Including a scalar field, the model is integrable in the case where one of the Einstein spaces is not Ricci-flat and the cosmological constant vanishes. Here, we discuss the case of nonzero cosmological constant, free massless scalar field and all spaces $M_i, i = 0, \ldots, \alpha$, being Ricci-flat (for details and references see [4, 11]).

The formal Lagrangian reads

$$L = \frac{1}{2} \sum_{i,j=0}^{\alpha} (m_{ij} \dot{\xi}^i \dot{\xi}^j + \dot{\varphi}^2) - \Phi \ , \tag{20}$$

with the energy constraint imposed

$$E = \frac{1}{2} \sum_{i,j=0}^{\alpha} (m_{ij} \dot{\xi}^i \dot{\xi}^j + \dot{\varphi}^2) + \Phi = 0 \ . \tag{21}$$

The potential is given by

$$\Phi = \Phi[\xi, \varphi] = \exp(2\sum_{i=0}^{\alpha} d_i \xi^i) \left(-\frac{1}{2} \sum_{j=0}^{\alpha} d_j (d_j - 1) k_j e^{-2\xi^j} + U[\varphi] + \Lambda \right) \ . \tag{22}$$

At the quantum level the constraint (21) is modified into the WDW equation

$$\left(\frac{1}{2} \left(G^{ij} \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \xi^j} + \frac{\partial^2}{\partial \varphi^2} \right) - \Phi[\xi, \varphi] \right) \Psi[\xi, \varphi] = 0 \ , \tag{23}$$

where $\Psi = \Psi[\xi, \varphi]$ is the wave function of the universe. The minisuperspace metric can be diagonalized using special coordinates [23].

It is usually assumed that on Planck scale processes with topology changes should take place. For this reason Hawking and Page introduced the notion of quantum wormholes as a quantum extension of the classical wormhole paradigma. They defined the quantum wormholes as solutions of the Wheeler-DeWitt (WDW) equation with the following boundary conditions [20]:

(i) the wave function is exponentially damped for large spatial geometry,

(ii) the wave function is regular when the spatial geometry degenerates.

In the case of our Wheeler-deWitt equation corresponding calculations have been made in [4]. In 4-dimensional models with the scale factor having a turning point (at the minimum) the production of the Lorentzian space-time may be treated as a quantum tunneling process ("birth from nothing"). The universe appears spontaneously going through the potential barrier with the size equal to the size of the Lorentzian universe at the turning point. But in multidimensional case the situation becomes more complicated. The factor spaces M_i may reach their minimum points at different times. The "birth from nothing" for each factor space takes place at a different value of time. If the difference between these events is large enough the extra dimensions may be unobservable because they are hidden from us by a potential barrier.

Further inside into the solutions to multifactor cosmological models can be found using the conformal equivalence between different models [25]. For Lagrangians $L[R, \varphi]$ depending only on Ricci curvature R and a scalar field φ, there exists an explicit description of conformal equivalence, with the minimal coupling model and the conformal coupling model as particular representatives of a conformal class [10]. The domains of equivalence are separated by certain critical values of the scalar field φ. Furthermore the coupling constant ξ of the coupling between φ and R is critical at both, the minimal value $\xi = 0$ and the conformal value $\xi_c = \frac{D-2}{4(D-1)}$.

For vanishing potential of the minimally coupled scalar field one finds a multidimensional generalization of Kasner's solution. Its scale factor singularity vanishes in the conformal coupling model. Static internal spaces in the minimal model become time-dependent in the conformal one. The nonsingular conformal solution has a particularly interesting region, where internal spaces shrink while the external space expands. While the Lorentzian solution relates to a creation of the universe at finite scale, it's Euclidean counterpart is an (instanton) wormhole.

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