GRAVITATIONAL FIELDS WITH NULL POINTS OF THE DETERMINANT OF $g_{\mu\nu}$

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Abstract. In the quantum field theory the existence and the structure of the causality cone (light cone) is of decisive importance. It has been noted severally that the divergences in the quantum theory of interacting fields can be eliminated through the assumption that the light cone in a region of extremely strong fields either is distorted or does not exist at all [cannot be defined] (nonlocal field theories, field theories with an elementary length). - In Einstein's geometric theory of gravitation the structure of the causality cone is defined by the gravitational field $g_{\mu\nu}$. Nevertheless, according to the usual interpretation of the general theory of relativity, in an infinitesimal region of space the light cone is identical with the Minkowskian null cone. It is in fact required that everywhere (also in the region of a strong gravitational field) the metric possess the Lorentz-Minkowski signature $s = -2$, so that on an infinitesimal scale one can always have $g_{\mu\nu} = \eta_{\mu\nu}$. A nonMinkowskian signature (in particular a definite one) in the interior region of strong fields, due to the Minkowskian signature of the exterior, would require the vanishing of the determinant of $g_{\mu\nu}$ on a timelike hypersurface. - It will be shown here that such null points of finite order can be entirely compatible with the vacuum equations $R_{\mu\nu} = 0$ in the sense of a limit. - Then static gravitation fields are considered, and for them from $R_{\mu\nu} = 0$ it is derived that for $\det g_{ik} \neq 0$ the null points of $\det g_{\mu\nu}$ must be of the second order. An example for this is the bridge model of the Schwarzschild metric by Einstein and Rosen.

1. The causality cone and the signature of the gravitational field $g_{\mu\nu}$

In the general theory of relativity it is usually required in general that the special theory of relativity be valid in infinitesimal regions of the four-dimensional space V_4. From here it follows that, in a normal-geodesic coordinate system around the considered point P, one can always obtain $(g_{\mu\nu})_P = \eta_{\mu\nu}$. Here $\eta_{\mu\nu}$ is the Minkowskian fundamental tensor

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{pmatrix}.$$

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According to Sylvester's theorem of inertia[1], then the Riemannian V_4 of the general theory of relativity has everywhere the same index of inertia

$$(-1, -1, -1, +1)$$

as the universe of Minkowski, and the signature $s = -3 + 1 = -2$.

This Lorentz-Minkowski signature of V_4 means physically that at each point P of V_4 a causality cone (light or null cone) exists, that in the infinitesimal neighbourhood of the point P coincides with the Minkowskian cone at P. In a sufficiently small region of the Riemannian spacetime universe V_4 all the causality relations are therefore valid in the form in which they are valid in the special theory of relativity. - In a finite region, on the contrary, due to the existence of a gravitational field the lightcone of P behaves otherwise than in the flat Minkowski universe, and the global properties of the causality cone in the region of a strong gravitational field can be totally different from the ones of the Minkowskian null cone[2]. Hence on a large scale, due to the existence of the gravitational field, the causal connection can have a form totally different from the one assumed in the special-relativistic quantum field theories. This occurs when one deals with the regions in which the wave fields become extremely strong, and as a consequence, according to Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu} \tag{1}$$

the gravitational field too eventually becomes extremely strong.

If the coordinate system is so chosen that from the vanishing of the Riemann tensor $R_{\sigma\mu\nu\lambda}$ it follows also $g_{\mu\nu} = \eta_{\mu\nu}$, in regions in which $g_{\mu\nu} - \eta_{\mu\nu}$ is of the order of magnitude 1, one must expect in general a quite unusual global behaviour of the causality cone[3]. This has on occasion far reaching consequences for the properties of the generalised Green's functions, hence for the causality relations. Furthermore, a null cone that globally differs much from the Minkowskian one can lead to a temporal asymmetry of spacetime and to analogous topological consequences. Also the character of the isometry groups could change. All this leads to new questions when dealing with the elementary particle problem in the framework of the particle program of Einstein, or of the geometrodynamics of Wheeler (see below footnote 7).

It seems however that the existence of light cones at any point P, chosen at will, *i.e.* the infinitesimal validity of the Minkowski metric also in a region of very strong fields, is indissolubly and intrinsically linked to the appearance of the divergences in the quantum field theory. All the singularities of the quantum field theories happen just on light cones.

Ad hoc theories have been proposed for the elimination of the difficulties associated with the existence of the light cone also in a region of extremely

[1] see e.g. L. P. Eisenhart, Riemannian Geometry, Princeton 1949, p. 23.

[2] see on this point D. Finkelstein, Physic. Rev. **110**, 965 (1958); A. Papapetrou and H. Treder, Ann. Physik (7) **6**, 311 (1960).

[3] see previous footnote.

strong fields. Either it is assumed an incomplete localisability of the fields due to the existence of an elementary length l, or the Green functions are altered through the introduction of structure functions.

Yukawa once proposed to assume that inside the nucleus of the elementary particle (hard core) the retarded function of special relativity does not intervene, but that all the interactions occur instantaneously. This means just that according to Yukawa in the "core of the elementary particle" no light cone exists, hence the metric is definite.

Now, Einstein's geometric theory of gravitation is evidently competent for assertions about the metric. In fact Einstein's equations (1) are a system of partial differential equations for the metric tensor $g_{\mu\nu}$. It has been already observed severally that a consistent fusion of quantum theory and gravitation theory *per se* leads to an elementary length. In fact from Newton's gravitational constant f (connected with Einstein's constant κ according to $\kappa = 8\pi f/c^4$), from c, the velocity of light in an inertial system, and from Planck's constant h, Planck's elementary length

$$l_{Planck} = \sqrt{\frac{hf}{c^3}} \approx 10^{33}\text{cm} \tag{2}$$

can be formed. This elementary length actually appears as a limit to the measurability of lengths in a quantum theory of the gravitational field[4].

The usual standpoint of the general theory of relativity is however the one outlined above. In fact in the world regions, in which the gravitational fields are weak, hence everywhere except for very small regions in proximity to the quantum theoretical singularities in the interior of the particles, the $g_{\mu\nu}$ (in a quasi-Cartesian system of coordinates) differ only slightly from $\eta_{\mu\nu}$. In these regions it holds

$$\det g_{\mu\nu} \equiv |g_{\mu\nu}| \equiv g < 0. \tag{3}$$

The condition (3) for the determinant allows for the metric of V_4 only the signatures

$$s = +2 \tag{4}$$

or

$$s = -2. \tag{5}$$

(4) corresponds to the existence of 3 independent timelike and one spacelike directions at any point P, hence to the index of inertia

$$(+1, +1, +1, -1). \tag{6}$$

(5) corresponds to the existence of 3 independent spacelike and one timelike directions in P, hence to the Minkowskian index of inertia[5]

$$(-1, -1, -1, +1). \tag{7}$$

[4] see J.A. Wheeler, Annals of Physics **2**, 604 (1957); H. Treder, Monatsberichte der Deutschen Akademie der Wissenschaften zu Berlin **3**, 241 (1961).

[5] It is assumed the usual sign convention of the general theory of relativity.

But since in the exterior region with weak gravitational field it holds just (7), for reasons of continuity the signature (5) shall hold everywhere, if (3) is everywhere required.

Now, a change of sign of the determinant g brings for reasons of continuity to the existence of a hypersurface S (that on occasion can also degenerate to a surface, line or point), at which it holds

$$g_S = 0. \tag{8}$$

A null point of g on S derives from the fact that either

a) a direction, spacelike for $g < 0$, or
b) a direction, timelike for $g < 0$,

degenerates to a null direction.

When g has on S a null point of odd order, if on the left side of S it is $g < 0$, on the right side it shall be $g > 0$. According to whether

a) an originally spacelike direction becomes timelike for $g > 0$, or
b) an originally timelike direction becomes spacelike,

on the right side of S the indices of inertia are different[6].
In case a) one has

$$(-1, -1, +1, +1) \quad \text{with} \quad s = 0, \tag{9}$$

and in case b)

$$(-1, -1, -1, -1) \quad \text{with} \quad s = -4. \tag{10}$$

In the first case, on a second hypersurface S^* (with $g_{s^*} = 0$), another spacelike direction might become timelike, so that on the right side of S^* the index of inertia reads

$$(-1, +1, +1, +1) \quad \text{with} \quad s = +2. \tag{11}$$

On the right side of S^* the metric is therefore on an infinitesimal scale again of Minkowskian type, however with the mirror index of inertia (11).

After what has been said, the case (10) appears physically important. We can in fact assume the hypersurface S to be timelike (see below §3) and closed in two dimensions. Then the region surrounded by S (world tube) has at all times the signature (10), while outside S the signature persists in being Minkowskian. One could try to interpret the internal region as "history of the core of an elementary particle". However, since there the metric is negative definite, in that region all the field equations, inclusive of the Einstein equations, are elliptic, so that there, according to the requisite of Yukawa, one has an instantaneous and not retarded interaction. The (four-dimensional) internal region with the negative definite metric will have as border the hypersurface S, at which the boundary condition $g = 0$ must be satisfied. Due to the elliptic character of Einstein's equations in the inner region this boundary value problem leads by itself to an eigenvalue problem,

[6]see *e.g.* A.Z. Petrow, Einstein-Räume, Moskau 1961.

hence to a discrete multeplicity of solutions of Einstein's equations (1), to which one will perhaps additionally impose cylindrical symmetry.

However already the existence of hypersurfaces S, at which (8) holds, but g has a root of even multiplicity, would be of physical interest, because then at least at the points of S the light cone might not exist. This will be the case when (see §3) the timelike direction degenerates to a null direction (case b). The metric at S will be semidefinite with the index of inertia

$$(-1, -1, -1,\ 0) \quad \text{and} \quad s = -3. \tag{12}$$

If instead (case a) a spacelike direction degenerates to an isotropic one, at S we have

$$(0, -1, -1, +1) \quad \text{and} \quad s = -1, \tag{13}$$

hence an indefinite metric at S with lightcones, that at S degenerate to two-dimensional ones[7].

2. The field equations $g^2 R_{\mu\nu} = 0$

In the general theory of relativity the null points of g will be in general considered as field singularities. They can not be eliminated by any regular transformation of coordinates. In particular field singulatities shall always appear, when the null points can not be consistently interpreted as a result of irregular transformations.

This idea stems from the fact that from (8) it follows that some components of the contravariant tensor $g^{\mu\nu}$ become infinite at S, since it is

$$\det g^{\mu\nu} = \frac{1}{g}. \tag{14}$$

However on this point Einstein[8] remarked that no direct geometric meaning is in general associated to the contravariant $g^{\mu\nu}$. In the Riemannian geometry the infinitesimal length is defined through the covariant $g_{\mu\nu}$, namely through the line element

$$\mathrm{d}s = \sqrt{g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}.$$

The $g^{\mu\nu}$ enter instead both the Riemann tensor $R^{\sigma}_{\mu\nu\lambda}$ and the Einstein-Ricci tensor $R_{\mu\nu}$, so that at surfaces S, at which (8) holds, $R_{\mu\nu}$ becomes an evidently nonsensical expression, since $R_{\mu\nu}$ is bilinear in the $g^{\mu\nu}$, hence in

[7] A problem connected with the previous questions is whether, due to the existence of a Killing vector (or of an analogous but weaker symmetry of the gravitational field), the quotient space V_3^* can change its signature, without an overall change of the Minkowski signature of V_4. In a work of A. Papapetrou and the present author it has been investigated in particular whether regular gravitational fields do exist with an asymptotic timelike Killing vector, that in the region of strong fields becomes isotropic and then spacelike.

[8] A. Einstein and N. Rosen, Physic. Rev. **48**, 73 (1935); **49**, 404 (1936); A. Einstein, Jour. Franklin Inst. **221**, 313 (1936).

$1/g$. Therefore Einstein writes the field equation for the vacuum not in the form

$$R_{\mu\nu} = 0; \tag{15}$$

he writes instead

$$g^2 R_{\mu\nu} = 0. \tag{16}$$

The tensor density of fourth degree $g^2 R_{\mu\nu}$ is evidently everywhere regular when the $g_{\mu\nu}$, together with all the first and second derivatives that enter $R_{\mu\nu}$, are regular. To this end it is sufficient that the $g_{\mu\nu}$ and their first derivatives be continuously differentiable.

Therefore, at the left-hand side of (16) there is a well defined expression also at points where (8) holds. Hence (16) could also possess solutions for which the sign of g changes at a hypersurface S, and as a consequence the signature s of the Einsteinean V_4 is different on both sides of S, or more generally, solutions for which g vanishes on S.

However, if the modified vacuum field equations (16) are satisfied on S and in a layer $S' < S < S''$ of finite width, in this layer the original Einstein's vacuum field equations (15) hold too, and in particular, in the sense of a limit, also on S: solutions of the modified field equations (16) of Einstein, for which at S the determinant g has a null point of n-th order, are (in the limit) also solutions of (15).

In fact, from the validity of (16) in a finite layer $S' < S < S''$, it follows that on S, besides the density $g^2 R_{\mu\nu}$, all the derivatives of $g^2 R_{\mu\nu}$ vanish too. If we then give to the hypersurface S the equation

$$x^1 = 0, \tag{17}$$

it follows:

$$\begin{aligned}(g^2 R_{\mu\nu})_{x^1=0} &= \left(\frac{\partial}{\partial x^1}\,(g^2 R_{\mu\nu})\right)_{x^1=0} = \cdots = \left(\frac{\partial^n}{(\partial x^1)^n}\,(g^2 R_{\mu\nu})\right)_{x^1=0} \\ &= \left(\frac{\partial^{n+1}}{(\partial x^1)^{n+1}}\,(g^2 R_{\mu\nu})\right)_{x^1=0} = \cdots = 0.\end{aligned} \tag{18}$$

If we then form for $x^1 \to 0$, *i.e.* for $g \to 0$, the limit value

$$\lim_{x^1 \to 0} \left(\frac{g^2 R_{\mu\nu}}{g^2}\right) = (R_{\mu\nu})_{x^1=0}, \tag{19}$$

according to L'Hôpital's rule, from (18) follows:

$$\lim_{x^1 \to 0} \, (R_{\mu\nu}) = 0. \tag{20}$$

Then, for null points of g of finite order, together with (16), Einstein's vacuum equations (15) are satisfied too. Regular Einstein spaces with g

vanishing at a hypersurface do exist, when there are solutions of (16) for which g vanishes at a hypersurface.

In the following we deal with the simplest particular case of a gravitational field $g_{\mu\nu}$ with a determinant g that vanishes at a hypersurface S. For this problem, dealt with in the following §3, one can calculate directly from (15), and it is not necessary to execute explicitly the limit (19).

3. Static gravitational fields with a hypersurface $g = 0$

We will now assume that a one-dimensional isometry group exists, so that in an adapted coordinate system the "cylindrical condition"

$$g_{\mu\nu,4} = 0 \tag{21}$$

can be satisfied. The coordinate x^4 can be either spacelike or timelike; however, it cannot be isotropic. Furthermore we prescribe that the field be "static" with respect to the coordinate x^4, *i.e.* we suppose that

$$g_{i4} = 0 \tag{22}$$

holds too[9]. Then it is

$$g = |g_{ik}| g_{44} \tag{23}$$

and

$$g_{il} g^{kl} = \delta_i^k, \ g^{i4} = 0, \ g^{44} = \frac{1}{g_{44}}. \tag{24}$$

The hypersurface S must be independent of x^4 too, *i.e.* S must contain the coordinate lines of x^4, so that the isometry exists globally. We can hence [by means of a transformation $\overline{x}^1 = \overline{x}^1(x^i)$][10] reduce S to the form (17).

g vanishes on S either when

$$|g_{ik}|_{x^1=0} = 0, \tag{25}$$

or when

$$(g_{44})_{x^1=0} = 0. \tag{26}$$

Now we assume that the subspace V_3 defined by x^4=const. possesses a nondegenerate metric $\mathrm{d}s^2 = g_{ik}\mathrm{d}x^i\mathrm{d}x^k$. Then it is everywhere

$$|g_{ik}| \neq 0,$$

and, when (8) holds, (26) must hold too.

Through a transformation

$$\overline{x}^2 = \overline{x}^2(x^i), \quad \overline{x}^3 = \overline{x}^3(x^i), \quad \overline{x}^1 \equiv x^1,$$

that does not spoil the already achieved simplifications, we can further obtain

$$g_{12} = g_{13} = 0. \tag{27}$$

[9] The Greek indices shall run from 1 to 4, the Latin ones from 1 to 3.

[10] see A. Papapetrou and H. Treder, Ann. Physik (7) **6**, 311 (1960).

The matrix of g_{ik} thus takes the form:

$$g_{ik} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & g_{23} \\ 0 & g_{23} & g_{33} \end{pmatrix}, \tag{28}$$

and the contragredient one respectively reads

$$g^{ik} = \begin{pmatrix} g^{11} & 0 & 0 \\ 0 & g^{22} & g^{23} \\ 0 & g^{23} & g^{33} \end{pmatrix}. \tag{29}$$

Furthermore, it is

$$g^{11} = \frac{1}{g_{11}} \neq 0, \tag{30}$$

since g_{11} is everywhere limited.

From (30) it follows that the vector normal to S is not isotropic, despite the fact that the interior metric of S has the degenerate form

$$\mathrm{d}\sigma^2 = g_{22}(\mathrm{d}x^2)^2 + 2g_{23}\mathrm{d}x^2\mathrm{d}x^3 + g_{33}(\mathrm{d}x^3)^2.$$

This is obviously a consequence of the vanishing of the overall determinant g on S. - From (24) we infer that the only field component that has a pole on S is g^{44}.

In the neighbourhood of S we write g_{44}, expanded in powers of x^1 and, by taking (26) into account[11]:

$$g_{44} = \underset{n}{\alpha}\,(x^2, x^3)(x^1)^n + \underset{n+m}{\alpha}\,(x^2, x^3)(x^1)^{n+m} + \cdots, \tag{31}$$

with n and $m > 0$. It is not necessary that (31) be the beginning of a Taylor series, and we do not require a priori that n and m be natural numbers. It is only necessary that the first and second derivatives do exist. From the fact that $g_{44,1}$ is limited follows in particular that it must be $n \geq 1$. For g, because of (23) and of $|g_{ik}| \neq 0$, an analogous expression holds:

$$g = (|g_{ik}|)_{x^1=0}\ \underset{2}{\alpha}\,(x^2, x^3)(x^1)^n + \underset{n+m}{A}\,(x^2, x^3)(x^1)^{n+m} + \cdots. \tag{32}$$

We will assume that for $x^1 > 0$ (the "external" region) it is $g_{44} > 0$ too, hence x^4 in the outer region is a timelike coordinate[12]. Let us pose for brevity $g_{44} = V^2$ and $-|g_{ik}| = \gamma^2$. - For $x^1 = 0$ here it is $g^{11} < 0$, hence the surface $x^1 = 0$ is timelike. In its points no null cone exists, because for $x^1 = 0$ there is only one null direction, the x^4 direction. This fact results

[11]The null points at $x^1 = 0$ cannot be removed by any regular transformation.

[12]As it can be easily seen, the assumption $g^{11} > 0$, $g_{44} < 0$ (for $x^1 > 0$) does not produce, with respect to the structure of $g_{\mu\nu}$, results different from the ones that will be derived here.

immediately from the observation that, because $|g_{ik}| \neq 0$, for $x^1 = 0$ g is of rank 3 and the matrix of g_{ik} is definite.

The field equations (15) read:

$$R_{44} = \frac{V}{\gamma}\left(g^{ik}\gamma V_{,i}\right)_{,k} = 0, \tag{33}$$

$$R_{i4} = 0, \tag{34}$$

$$R_{kl} = P_{kl} + \frac{1}{V}\left(V_{,kl} - \Gamma^i_{kl} V_{,i}\right) = 0. \tag{35}$$

Here P_{ik} is the Ricci tensor, formed with g_{ik}, of the usual V_3 defined by $x^4 = 0$, and consequently is everywhere limited. Also the Γ^i_{kl} contain only the g_{ik} and g^{mn}, and therefore are everywhere limited too.

Due to $\gamma \neq 0$, in the neighbourhood of $x^1 = 0$, from (33) and from the expression (31) for g_{44} follows[13]:

$$\begin{aligned} V\left[\gamma g^{11} V_{,11} + \left(\gamma g^{11}\right)_{,1} V_{,1}\right] = &\left[\underset{n}{\alpha}^{\frac{1}{2}}\left((x^1)^{\frac{n}{2}} + \cdots\right)\right] \times \\ &\left[\left((\gamma g^{11})_{x^1=0}\left(\frac{n}{2}\left(\frac{n}{2} - 1\right)\right)(x^1)^{\left(\frac{n}{2}-1\right)} + \cdots\right)\right] \\ &+ \left[\underset{n}{\alpha}^{\frac{1}{2}}\left((x^1)^{\frac{n}{2}} + \cdots\right)\right] \times \\ &\left[\left((\gamma g^{11})_{,1}\right)_{x^1=0}\left(\frac{n}{2}(x^1)^{\left(\frac{n}{2}-1\right)} + \cdots\right) + \cdots\right] = 0. \end{aligned} \tag{36}$$

All the powers of x^1 must vanish individually. Since $g^{11}\gamma \neq 0$, the term with the lowest power of x^1 is in general the term

$$\left(V\gamma g^{11} V_{,11}\right)_{x^1=0}. \tag{37}$$

Therefore (37) always tends to zero more slowly than the next term

$$\left(V(\gamma g^{11})_{,1} V_{,1}\right)_{x^1=0}, \tag{38}$$

provided that V tends to zero more quickly than x^1. Hence in order that for $x^1 \to 0$ (33) may be satisfied [and also (31) may be valid], V must possess just a nullpoint of first order. Then it must be $n = 2$, *i.e.* one must have

$$V = \underset{2}{\alpha}^{\frac{1}{2}}(x^2, x^3)(x^1)^1 + \cdots \tag{39}$$

[13]One observes that, for $n \geq 2$, R_{44} at $x^1 = 0$ remains limited and therefore meaningful.

and hence

$$g_{44} = \underset{2}{\alpha}\,(x^2, x^3)(x^1)^2 + \cdots, \tag{40}$$

where by necessity the coefficient of $(x^1)^2$,

$$\underset{2}{\alpha}\,(x^2, x^3),$$

is different from zero. - g_{44} and hence g have at S, in general, a null point of second order:

$$g = |g_{ik}|_{x^1=0}\,\underset{2}{\alpha}\,(x^2, x^3)(x^1)^2 + \cdots. \tag{41}$$

We can evaluate also the next term in the expansion (31) for g_{44}. As we will see immediately (see below), (35) requires in fact that all the $g_{ab,1}$ vanish for $x^1 = 0$ at least like x^1. If furthermore $g_{11,1}$ vanishes too, like in §4, with (39) one gets

$$(\gamma g^{11})_{,1} V_{,1} = \beta(x^2, x^3)x^1\,\underset{4}{\alpha}^{\frac{1}{2}}\,(x^2, x^3) + \cdots. \tag{42}$$

For (33) to be satisfied, one needs that for $V_{,11}$ it holds:

$$V_{,11} = v(x^2, x^3)x^1 + \cdots. \tag{43}$$

Then (39) specialises to

$$V = \underset{2}{\alpha}^{\frac{1}{2}}\,(x^2, x^3)x^1 + \underset{3}{\alpha}\,(x^2, x^3)(x^1)^3 + \cdots, \tag{44}$$

so that for the two lowest vanishing terms of g_{44} we find

$$g_{44} = \underset{2}{\alpha}\,(x^2, x^3)(x^1)^2 + \underset{4}{\alpha}\,(x^2, x^3)(x^1)^4 + \cdots. \tag{45}$$

- If in particular (31) is a Taylor series, one obtains for the Taylor coefficients

$$\underset{0}{\alpha} = \underset{1}{\alpha} = \underset{3}{\alpha} = 0, \quad \underset{2}{\alpha} \neq 0. \tag{46}$$

- The expansion for g is completely analogous.

The field equations (34) are identities. - In the field equations (35) P_{kl} is limited, hence for $x^1 \to 0$ also the terms

(47) $$\frac{1}{V} V_{;kl}$$

must remain limited. Now, according to (39) it is

(48) $$V_{,1} \to \underset{2}{\alpha}^{\frac{1}{2}}, \quad V_{,ab} \to 0, \quad V_{,2} V_{,3} \to 0;$$

(49) $$\frac{1}{V} \to \underset{2}{\alpha}^{-\frac{1}{2}} (x^1)^{-1}.$$

Therefore it must be for $x^1 \to 0$:

(50) $$\Gamma^i_{ab} V_{,i} \to \Gamma^1_{ab} V_{,1} \to \Gamma^1_{ab} \alpha^{\frac{1}{2}} \to 0(x^1),$$

where Γ^1_{ab} according to (49) must have a null point (at least) of the first order:

(51) $$\Gamma^1_{ab} = \underset{1}{\alpha}^{\frac{1}{2}}_{ab} (x^2, x^3) x^1 + \cdots$$

($\underset{1}{\alpha}_{ab}$ $= 0$ is possible too). Due to (27) equation (50) means:

(52) $$\Gamma^1_{ab} = \frac{1}{2} g^{11} g_{ab,1} = \underset{1}{\alpha}_{ab} (x^2, x^3) x^1 + \cdots.$$

Hence [in keeping with (36)], for the g_{ab} it must hold

(53) $$g_{ab} = \underset{0}{\alpha}_{ab} (x^2, x^3) + \underset{2}{\alpha}_{ab} (x^2, x^3)(x^1)^2 + \cdots.$$

(Some $\underset{2}{\alpha}_{ab}$ may vanish).

In the equations (48-53) the indices a and b run from 2 to 3, hence for $x^1 \to 0$ the field equations $R_{22} = R_{23} = R_{33} = 0$ are immediately satisfied. Equations

$$R_{11} = R_{12} = R_{13} = 0$$

require instead that, up to terms that vanish faster than x^1, it holds:

(54) $$V_{,11} - V_{,1}\Gamma^1_{11} = V_{,12} - V_{,1}\Gamma^1_{12} = V_{,13} - V_{,1}\Gamma^1_{13} = 0.$$

If one poses for V

$$V = \underset{1}{\overline{\alpha}}\,(x^2,x^3)x^1 + \underset{1+k}{\overline{\alpha}}\,(x^2,x^3)(x^1)^{1+k} + \cdots,$$

one sees immediately that it must be $k \geq 1$. From (54), when $x^1 \to 0$, it follows for $k = 1$:

$$V_{,11} = 2\,\underset{2}{\overline{\alpha}} = \frac{1}{2}g^{11}g_{11,1}\,\underset{1}{\overline{\alpha}}\,,$$

$$V_{,1a} = \underset{1}{\overline{\alpha}}_{,a} = \frac{1}{2}g^{11}g_{11,a}\,\underset{1}{\overline{\alpha}} \quad (a = 2,3). \tag{55}$$

Therefore one has $\underset{2}{\overline{\alpha}} = 0$ and hence $\underset{3}{\alpha} = 0$ when $g_{11,1}$ vanishes for $x^1 \to 0$, as it happens with the Einstein-Rosen metric dealt with in §4. - One further infers from (55) that, when g_{11} is constant for $x^1 \to 0$, the same thing occurs for $\underset{1}{\overline{\alpha}}$ and hence $\underset{2}{\alpha}$ too.

4. About the bridge model of a particle according to Einstein and Rosen

Through the form, derived in §3, of the terms that vanish to the lowest order for g_{44} and g_{kl}, it is guaranteed that for $x^1 \to 0$ Einstein's vacuum field equations are fulfilled, although g vanishes for $x^1 \to 0$. Now it has been determined already in a completely different way a static gravitational field with $\det g_{ik} \neq 0$, that for $x^1 \neq 0$ satisfies the vacuum equations without singularities, and whose determinant g vanishes for $x^1 = 0$. This field is the Schwarzschild metric in the coordinates of the 'bridge model of a particle" of Einstein and Rosen[14].

The line element of Einstein and Rosen results from the Schwarzschild line element in spherical polar coordinates

$$\mathrm{d}s^2 = -\left(\frac{1}{1-\frac{2m}{r}}\mathrm{d}r^2 + r^2((\mathrm{d}x^2)^2 + \sin^2 x^2(\mathrm{d}x^3)^2)\right) + \left(1 - \frac{2m}{r}\right)(\mathrm{d}x^4)^2 \tag{56}$$

through the substitution

$$(x^1)^2 = r - 2m. \tag{57}$$

[14]A. Einstein and N. Rosen l.c. in footnote 8; J.A. Wheeler, Rev. mod. Physics **33**, 63 (1961). - The singularity of g_{11} that appears in (56) for $r = 2m$ will be eliminated through the Einstein-Rosen transformation (57).

It reads

$$(58)\qquad \mathrm{d}s^2 = -4\left(2m + (x^1)^2\right)(\mathrm{d}x^1)^2 - \left(2m + (x^1)^2\right)^2\left((\mathrm{d}x^2)^2 + \sin^2 x^2(\mathrm{d}x^3)^2\right) + \frac{(x^1)^2}{2m + (x^1)^2}(\mathrm{d}x^4)^2.$$

The value of the determinant is

$$(59)\qquad g = -4\left(2m + (x^1)^2\right)^4 (x^1)^2 \sin^2 x^2.$$

It vanishes for $x^1 = 0$ due to the vanishing of g_{44}. For small x^1 it holds for g_{44} the Taylor series

$$(60)\qquad g_{44} = \frac{(x^1)^2}{2m}\left(1 - \frac{(x^1)^2}{2m} + \cdots\right).$$

Einstein remarked that by using as foundation the modified field equations (16) the metric (58) is an everywhere regular solution. But, according to our discussion in §§2 and 3, (58) must satisfy also the usual vacuum equations (15), in particular also for $x^1 \to 0$. We see on the basis of the conditions derived in §3 that this is in fact so. The $g_{\mu\nu}$ are diagonal and independent of time. As required in formula (45), g_{44} according to (60) has a null point of second order, and for the Taylor coefficients one has - in keeping with (60):

$$(61)\qquad \underset{2}{\alpha} = \frac{1}{2m} \text{ and } \underset{0}{\alpha} = \underset{1}{\alpha} = \underset{3}{\alpha} = 0.$$

Furthermore, in keeping with (53) we find for the $g_{kl,1}$

$$(62)\qquad g_{11,1} = -8x^1, \quad g_{22,1} = -8mx^1 \text{ and } g_{33,1} = -8mx^1 \sin^2 x^2.$$

Also for finite $x^1 \neq 0$ (58) is a regular solution of (15). Hence the $g_{\mu\nu}$ in (58) are regular for positive m[15] and, with a coordinate transformation that is regular for $x^1 \neq 0$ and finite, they are transformed back to the Schwarzschild metric (56). - The singularity of the g_{ii} for $x^1 \to \infty$ can be evidently interpreted as an irregularity of the coordinates. It can be eliminated by prosecuting, at a surface $r = a > 0$, the transformation (57), with a sufficient degree of continuous differentiability, by means of a transformation that for $r \to \infty$ brings to Cartesian coordinates.

At variance with the hypotheses done in the known theorems of Serini, Lichnerowicz[16] and others, the regular metric (58) possesses a nonEuclidean topology. To any point of Schwarzschild's space with $r > 2m$ correspond in fact 2 coordinate values of x^1. The V^3 (and also the V^4) is therefore a sort of two-sheeted Riemann surface with a branch point at $x^1 = 0$, *i.e.* for

[15]For negative m, g_{44} has instead a pole for $(x^1)^2 = 2m$. As remarked by Einstein (l.c. in footnote 8) the prescription "a particle corresponds to a regular V_4 in the form of a "bridge model" with a surface $g = 0$" produces therefore positive particle masses. Hence the Schwarzschild field of a negative mass can not be transformed into a regular metric.

[16]see A. Lichnerowicz, Théories relativistes del la gravitation et de l'electromagnétisme, Paris 1955.

$g = 0$. - According to Einstein and Rosen, physically one must mutually identify the two sheets $x^1 > 0$ and $x^1 < 0$. The surface $g = 0$ is then the "bridge" between the two identical sheets. This "bridge" represents the particle. The Einstein-Rosen particle has therefore no "particle interior", and represents a generalised point particle. This could be a reason why there is no mass spectrum. In fact (see §1) just the "particle interior" should bring to eigenvalue problems.

5. Final remarks

Our previous discussion of the simplest pure gravitational field with null-points of the determinant g shows that the general theory of relativity - apart from the fact that it provides assertions about the global structure of the causality cones - allows also to produce metrics in which, at certain points, no causality cone locally exists.

From this we further see that in the gravitational theory of Einstein it is not consequent to impose a priori further general mathematical conditions on the gravitational field besides the field equations (15) and the Galilean boundary conditions. The usual condition $g \neq 0$ (and $s = -2$) excludes just metrics that constitute without doubt an Einstein space, and that can be physically of great importance.

More generally, one should a priori allow for all the possible gravitational fields $g_{\mu\nu}$ that fulfill Einstein's gravitational equations and the boundary conditions. In particolar the gravitational field under consideration should not be restricted either through a topological injunction (like the prescription of Euclidean topology), or through hypotheses of differentiability (like the prescription that the $g_{\mu\nu}$ be of class C_1 or C_2 [piecewise higher]). - The interesting ideas[17] of Wheeler about a geometrodynamics stand in fact on the introduction of nonEuclidean topologies in the general theory of relativity. And our investigations over gravitational shock waves have shown[18] that through the injunction: the $g_{\mu\nu}$ are of class C_1 (piecewise higher) the physically interesting shock waves of first order become excluded, although Einstein's equations can be satisfied in the sense of a limit also for discontinuous $g_{\mu\nu,\lambda}$. - Eventually the injunction of continuity for the $g_{\mu\nu}$ appears superfluous[19]. In fact Einstein's equations, again applied in the sense of

[17]see C. Misner and J.A. Wheeler, Ann. of Physics **2**, 525 (1957); J. A. Wheeler, Nuovo Cimento Suppl. **7** (1960) and J.A. Wheeler l.c. in footnote 14.

[18]A. Papapetrou and H. Treder, Math. Nachrichten **20**, 53 (1959); H. Treder, Gravitative Stoßwellen, Berlin 1962.

[19]A. Papapetrou and H. Treder, Math. Nachrichten **23**, 371 (1961); H. Treder l.c. in footnote 18.

a limit, prescribe by themselves that the $g_{\mu\nu}$ (up to a coordinate transformation) must be continuous. It is therefore sufficient to impose piecewise continuity and piecewise continuous differentiability.

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