Space-time curvature and recession velocities

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The problem of interpretation of recession velocities reflects straightforwardly the curvature of space-time. In a recent article it was claimed that this problom would show that the General Relativity Theory had to and would overrule the Special Relativity Theory [2, 3]. This must be corrected. It is shown that the simplistic definition of the recession velocity as change in distance on a space of constant cosmological time yields in fact a pseudo-euclidean angle and that the simplest correct definition of the recession velocity fits perfectly with the SRT formula for the Doppler effect.

The observation of structures in the transparent (redshift less than 1000) universe usually does not require the explicit use of the curvature of space-time. In addition, most of the evolution of stars, galaxies and even large-scale structure can be studied with pure newtonian gravity in an expanding space. This is the reason why the curvature of space-time appears to be a geometric attribute only. However, this produces a curious uncertainty in the interpretation of the relation between redshift and recession velocity.

The model for a homogeneous and isotropic expansion is the metric

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2[t]\mathrm{d}\chi^2 + r^2[\chi](\mathrm{d}\theta^2 + \sin^2[\theta]\mathrm{d}\varphi^2)\ . \quad (1)$$

When we consider motion to and from the freely chosen center $\chi = 0$, it is restricted to a two-dimensional space-time with the line element

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2[t]\mathrm{d}\chi^2\ . \quad (2)$$

This is a curved space-time. Its geometry is completely analogous to that of a two-dimensional rotation figure in three-dimensional space. We can map it by a Mercator projection (cylindrical and conformal map) through use of the conformal time η which is given by $a[t]\mathrm{d}\eta = c\mathrm{d}t$. The metric is now

$$\mathrm{d}s^2 = a^2[\eta](\mathrm{d}\eta^2 - \mathrm{d}\chi^2)\ . \quad (3)$$

In a $\eta - \chi$ map, the lines χ =const represent world-lines without peculiar velocity, that is the world-lines of ideal galaxies. The lines which are linear in *eta* and χ are lines of fixed peculiar velocity.

The distant galaxies are observed at an instant of their past, when their world-line intersects the past light cone of the observer at the instant of observation. Let us fix the coordinate $\chi = 0$ to the observer, then a signal which starts at a galaxy with the co-moving (i.e. reduced for expansion) distance χ and is observed at $t = 0$ is emitted at t given by

$$\int_t^{t_0} \frac{c\mathrm{d}t}{a[t]} = \chi \quad (4)$$

The signal is redshifted by

$$\frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = 1 + z = \frac{a[t_{\text{observation}}]}{a[t_{\text{emission}}]} \quad (5)$$

How to contrast this formula with the Doppler effect of some recession velocity $v_{\text{recession}}$? From SRT, we expect

$$1 + z = \sqrt{\frac{c+v}{c-v}}\ . \quad (6)$$

Here, the confusion begins. What is the recession velocity? It has to be a relative velocity between observer and source. Increase in distance between source and observer per increase in time, is this the relative velocity? Which distance, which time? When we choose the distance on the space t =const= t_0, we obtain velocites that exceed the speed of light [2, 3]. When we choose the length of the path of the light signal, we obtain $1 + z = c/(c - v)$. Is SRT wrong, overruled by some mysterious property of GRT?

We have to use SRT properly. Change in distance by increase in time is not the relative velocity, SRT speaks about. Relative velocity is a function of the difference in the directions of two world-lines at the event of their intersection. The expression in the formula for the Doppler shift (Bondi's k factor [1]) is the logarithm of the pseudoeuclidean angle. This pseudoeuclidean angle is best represented in the language of projective geometry: The square of Bondi's k factor is the cross ratio between the two directions of the world-lines under consideration and the directions of the light signals through the intersection event [4].

What about directions of world-lines at distant events? We have to device a parallel transport of the directions to some common event. In a flat Minkowski space-time, this is trivial and mostly not mentioned at all. In a curved space-time (curved by gravitation) this is not trivial and needs consideration. In a curved space-time, the parallel transport depends on the path along the transport is perfomed. This is the defining property of curvature. Hence, there is no relative velocity of distant objects without prior choice of transport paths.

For the metric (3), the (geodesic) transport equations for a vector $[A^\eta, A^\chi]$ are obtained in the form

$$\mathrm{d}A^\eta + \frac{\mathrm{d}\ln[a]}{\mathrm{d}\eta}(A^\eta \mathrm{d}\eta + A^\chi \mathrm{d}\chi) , \quad (7)$$

$$\mathrm{d}A^\chi + \frac{\mathrm{d}\ln[a]}{\mathrm{d}\eta}(A^\chi \mathrm{d}\eta + A^\eta \mathrm{d}\chi) . \quad (8)$$

Space-like vectors remain space-like, time-like ones time-like. Hence, the transport of the direction of the world-line of an ideal galaxy will always produce time-like directions. Any correctly constructed relative velocity is less than the speed of light, independent of any distance of time lag. A time-like vector and its transport equations can be written in the form

$$[A^\eta, A^\chi] = \frac{U}{a[\eta]}[\cosh\psi, \sinh\psi] , \quad (9)$$

$$\mathrm{d}\psi = \frac{\mathrm{d}\ln[a[\eta]]}{\mathrm{d}\eta}\mathrm{d}\chi , \quad U = \text{const} . \quad (10)$$

The quantity ψ is the pseudo-euclidean angle with the world-lines of our ideal galaxies, and its hyperbolic tangens is the relative velocity of these galaxies or peculiar velocity (in units of the speed of light). The correctly constructed relative velocities are obtained through integration of the equation for $\mathrm{d}\psi$. The result depends on the path, because the integrand varies with time. If it does not, i.e. $a[t] \propto t$, the space-time is flat. This is Milne's cosmological model.

The simplistic definition $w = \chi\ \mathrm{d}a/\mathrm{d}t$ [2, 3] is not a velocity, but the pseudo-euclidean angle. The relative velocity found by transport on some space $t = t_0$ of common cosmological time is correctly given by $v = c\ \tanh[w] < c$. The paths on such a space $t = t_0$ are not geodesics. A geodesic between to events on such a space passes through the future, and a transport along a geodesic will yield a different result. In any of these cases, we cannot expect to observe these velocities. We only calculate them and have to expect that there is no internal contradiction: There is none.

As for observation, we have to transport the direction of the world-line of the source at the event of emission to the event of observation. If this is done along the path which the photon takes through the space-time, we obtain

$$\mathrm{d}\eta = \mathrm{d}\chi , \quad \mathrm{d}\psi = \mathrm{d}\ \ln[a] , \quad \psi = \ln[\frac{a_{\text{observation}}}{a_{\text{emission}}}] . \quad (11)$$

With $v = c\tanh\psi$, this is easily transformed into the relation (6).

The general expression for the Doppler effect,

$$\frac{\omega_A}{\omega_B} = \frac{k_m[A]u_A^m}{k_m[B]u_B^m} , \quad (12)$$

turns out to be the generalization of equation (6): When we can suppose that k_m is the tangent to a light-like geodesic world-line, so that with u_B^m transported to A, we obtain

$$\frac{\omega_A}{\omega_B} = \frac{k_m[A]u_A^m}{k_m[A]u_B^m[A]} . \quad (13)$$

For one spatial dimension, $k_m u^m = \sqrt{\frac{c+v}{c-v}}$ [1].

Summary

1. GRT does not overrule SRT, it tells how SRT has to be correctly applied in curved by gravitation space-times.

2. There is a simple definition of relative velocity which yields the usual relativistic Doppler formula.

3. There is no superluminal object through expansion in the universe. The universe itself has no expansion velocity, it has a rate of expansion only.

References

[1] Bondi,H. (1962): *Relativity and Common Sense*, Heinemann, London.

[2] Davis, T.M., Lineweaver,Ch.H. (2003): Expanding confusion: common misconceptions of cosmological horizons and the superluminal expansion of the universe, *Proc.Astr.Soc.Australia* **21**, 97-109 (astro-ph 0310808).

[3] Lineweaver,Ch.H., Davis,T.M. (2005): Misconceptions about the Big Bang, *Scientific American* **March 2005**, 36-45.

[4] Liebscher, D.-E. (1978): Derivation of Einstein's velocity addition theorem through use of the invariant double ratio, *Found.Phys.* **8**, 131-135.

[5] Liebscher, D.-E. (2005): *Cosmology*, Springer, Berlin, Heidelberg, New York.

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